Little Experiments on Algorithms

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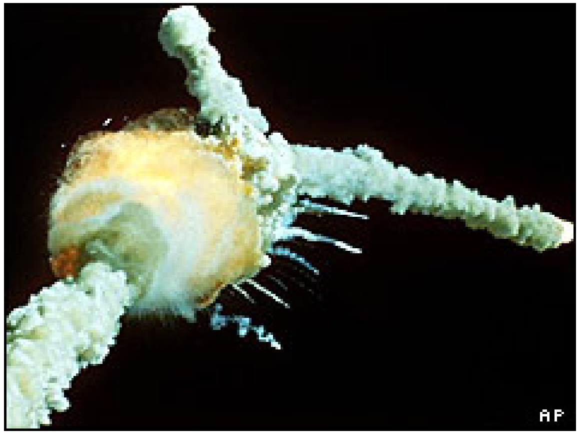
Outline

- Introduction
  - A Classic Tiny Experiment
  - Themes
- A Collection of Little Experiments
- Conclusions

Jumbo Engineering: The Challenger

A Big Project

A Big Problem



A Big Clue

O-rings

Their job: Expand so that no flames escape

Could they have failed?

A Tiny Experiment

Facts

The O-rings worked successfully in all previous launches

January 28, 1986, was a cold Florida day: 29°F

All earlier launches were on warmer days (≥53°F)

Hypothesis

O-ring expansion at 29°F
is substantially slower
than at higher temps

Testing the Hypothesis

The O-ring material, a C-clamp,
a glass of ice water, and
one Nobel laureate

Themes of this Workshop

*** 1. Extrapolation to Asymptopia

*** 2. Fair Comparisons

*** 3. Drawing Robust Conclusions

4. Data Overload — No problem is so big that it can't be run away from.

*** 5. Components of Running Time

6. Comparing Convergence Histories

7. Optimizing Parameter Settings — A little bit.

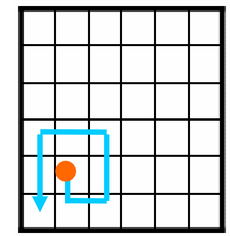
8. Sampling Streams

*** 9. Run Time Uncertainty

0. Experiment Design — My own axe to grind

The (Pretty) Good Old Days

Two paragraphs from Bentley, Weide and Yao, "Optimal Expected Time Algorithms for Closest Point Problems", *ACM TOMS* 6, 4, Dec. 1980:



2. Fair Comparison (to other algs)

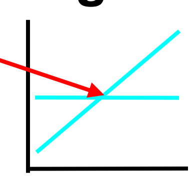
"Our Pascal implementation of the cell nearest neighbor algorithm required 11 lines to place the points in cells and 34 lines to search.

7. Parameter Setting: cell size (quite smooth)

The optimal number of points per cell was 3; densities ranging from 1 to 9, however, decreased the running time by only 10 percent.

1. (Casual) Extrapolation to Asymptopia

The average running time for nearest neighbor searching was the constant 2765 microseconds per search, on a PDP-KL10. This compares with 52*n* microseconds required by the linear search; the break-even point is at *n* = 53. To find all nearest neighbors in a 1000-point planar set, the linear-expected-time cell method required less than 2.8 seconds while the quadratic algorithm required 52 seconds.



3. Conclusions Robust across non-uniform inputs?

Cells, Cont.

3. Robust Conclusions

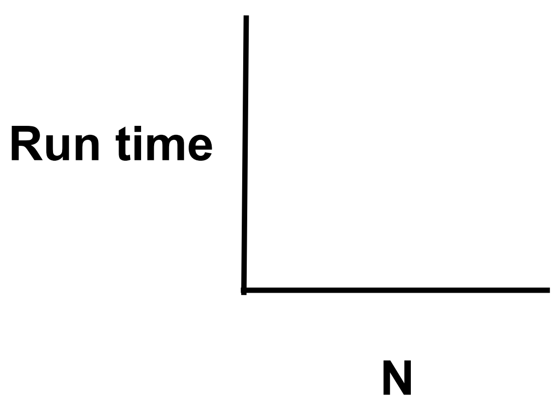
"We ran the program on two data sets representing the population centroids of political areas. The first set represented all census tracts in San Diego County and was fairly uniform over about half of a square; the second set represented all precincts in the (roughly square) State of New Mexico and was very clustered in the few large cities in the state."

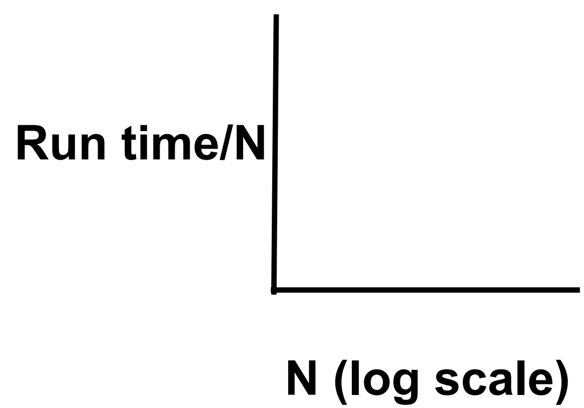
	San Diego		New Mexico	
Number of points	318		1122	
Quadratic algorithm				
Predicted secs	5.25	+2%	65.5	+2%
Observed secs	5.35		66.5	
Cell algorithm				
Predicted secs	0.88	+75%	3.11	+143%
Observed secs	1.40		7.56	
Optimum cell density	1.7		3.0	

7. Parameter Settings (no longer smooth)

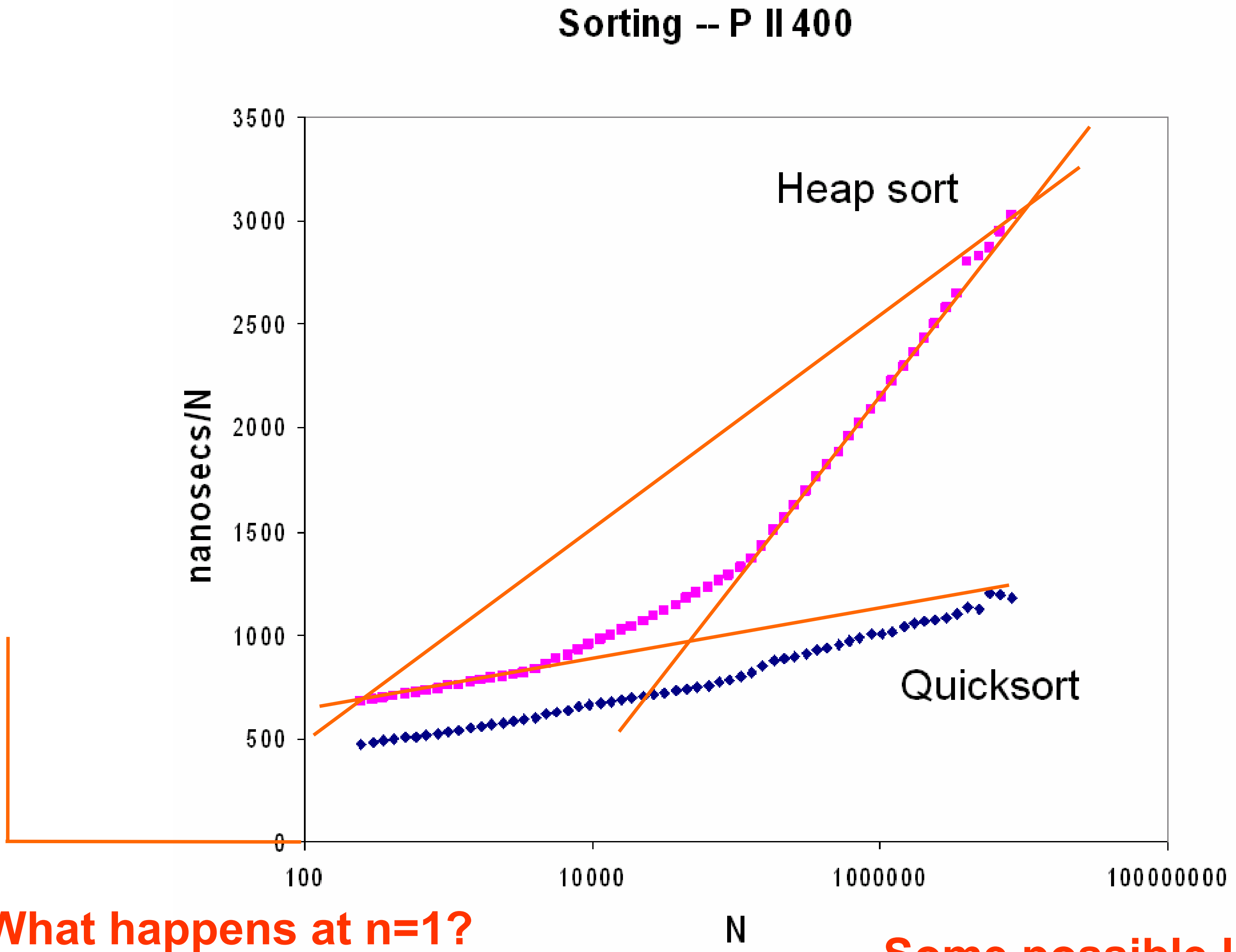
A Simpler Problem Today

- Which sort algorithm is faster: Quicksort or Heap sort?
  - Both are known to be $\Theta(N \lg N)$ for shuffled inputs
- Assume both have run times of $c\, N \lg N$
  - Run each at one value to determine each's constant
- Assume $c_1 N \lg N \;\; + \;\; c_2 N$
  - Run each at two values to determine both constants
  - Alternative: Plot a graph
    - What kind?





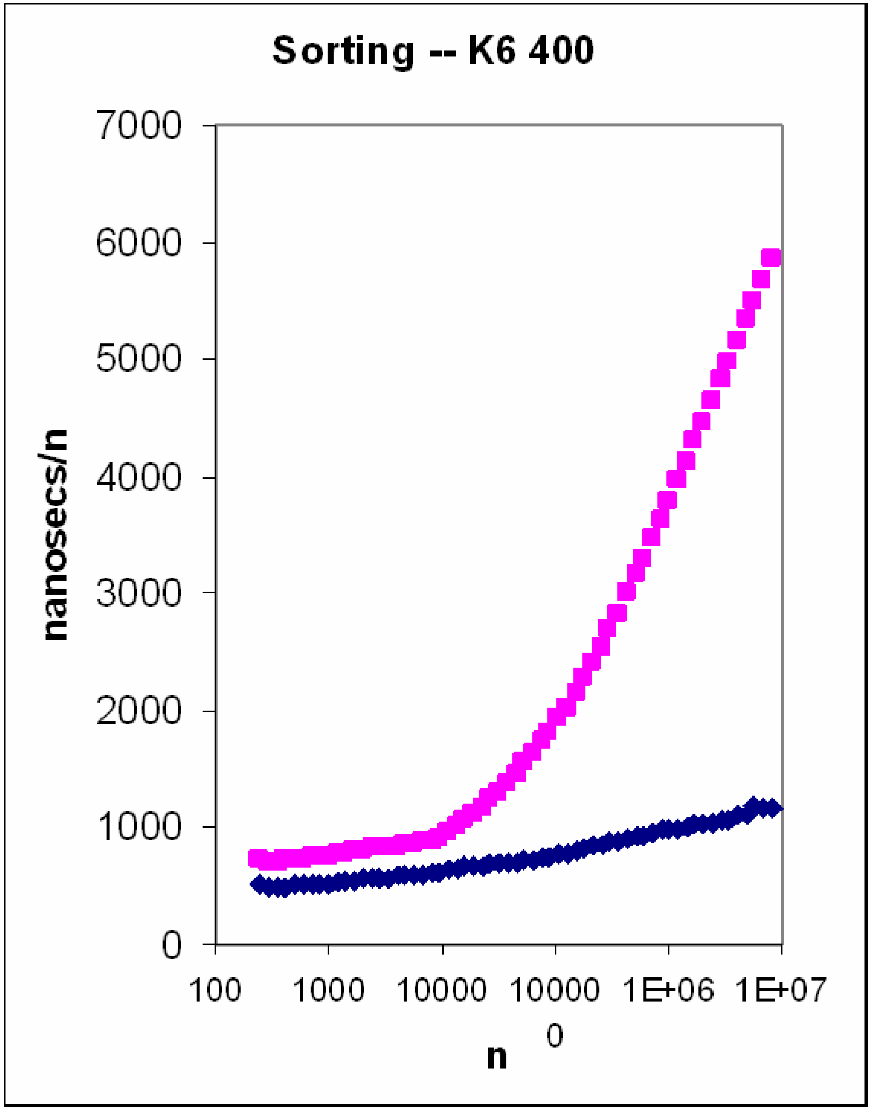
CPU Times for Sorting

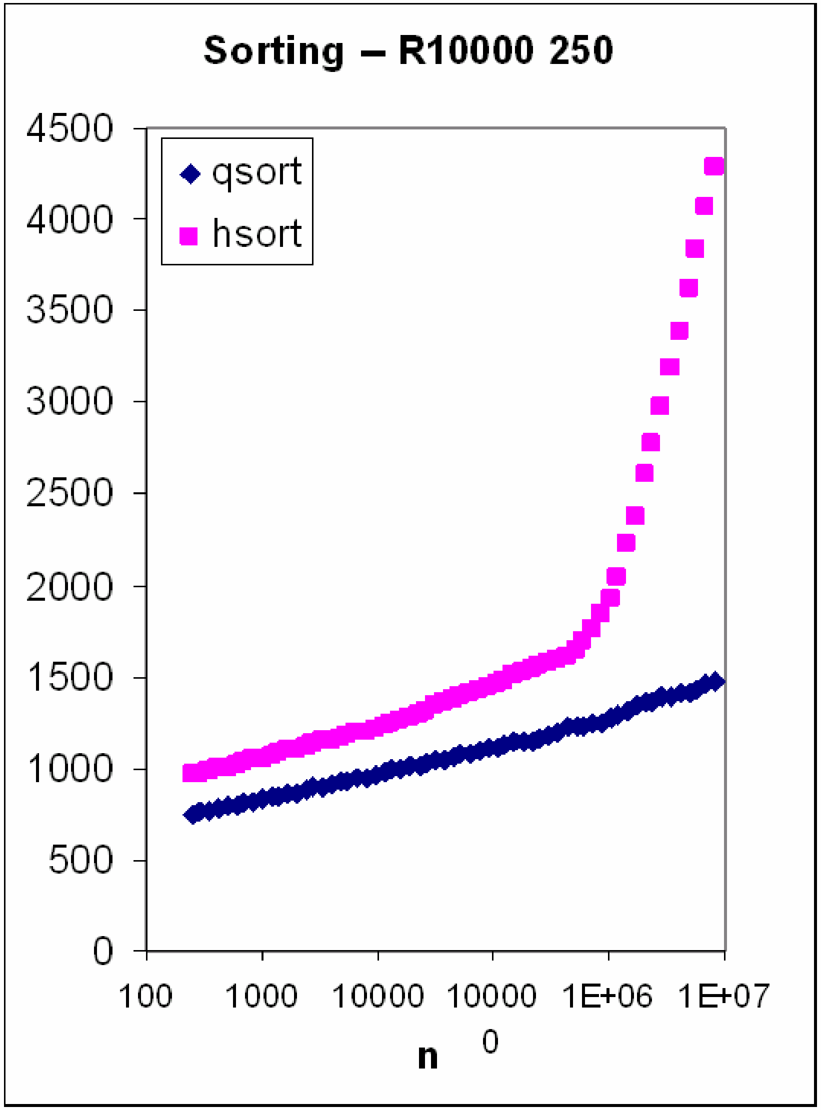


What happens at n=1?

Some possible lines

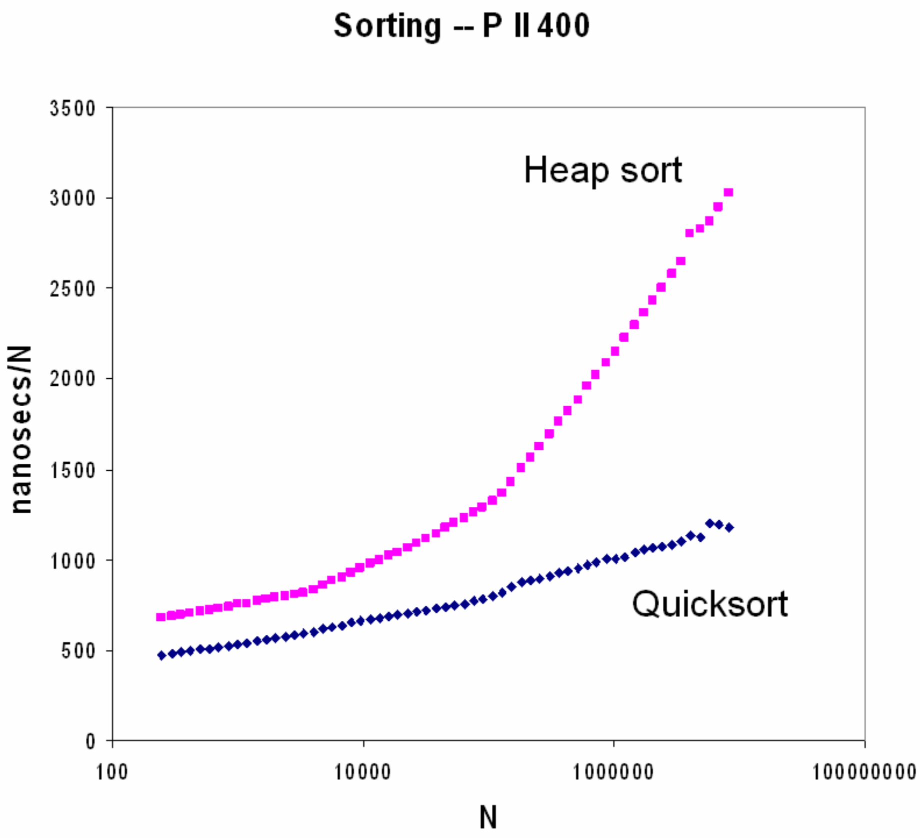
Sorting on Other Machines





Workshop Themes

1. Extrapolation to Asymptopia
    - Not there yet –
        - disk is right around the corner
2. Fair Comparisons
    - 3. Drawing Robust Conclusions
    - 5. Components of Running Time
9. Run Time Uncertainty
    - Why the hiccup near 10,000,000?
0. Experiment Design
    - Is there a better way?



A Cost Model for Memory

Goal: A little experiment to estimate memory costs

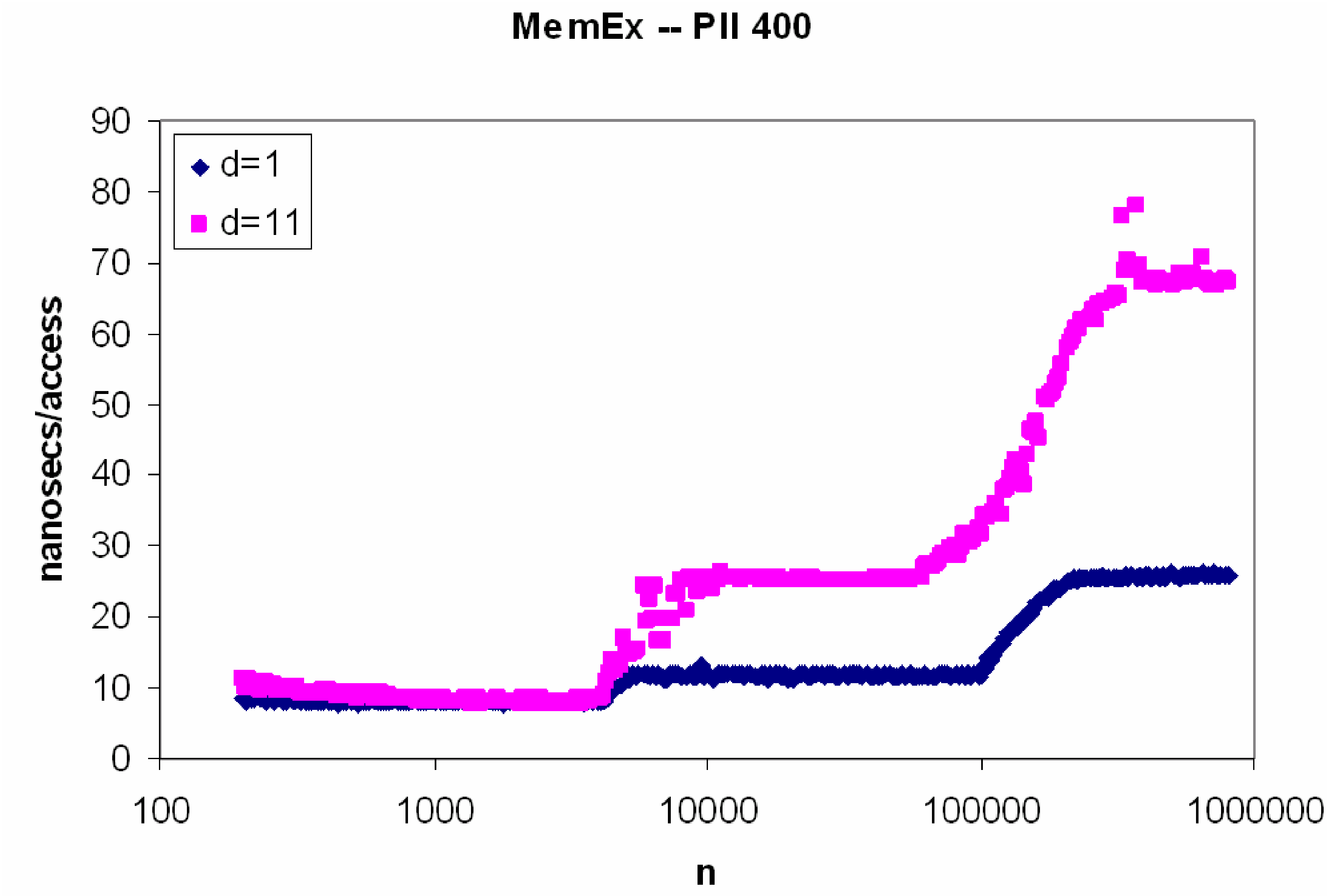
- Remove sorting to get to the essence of caching

The critical loop (n is array size, d is delta)

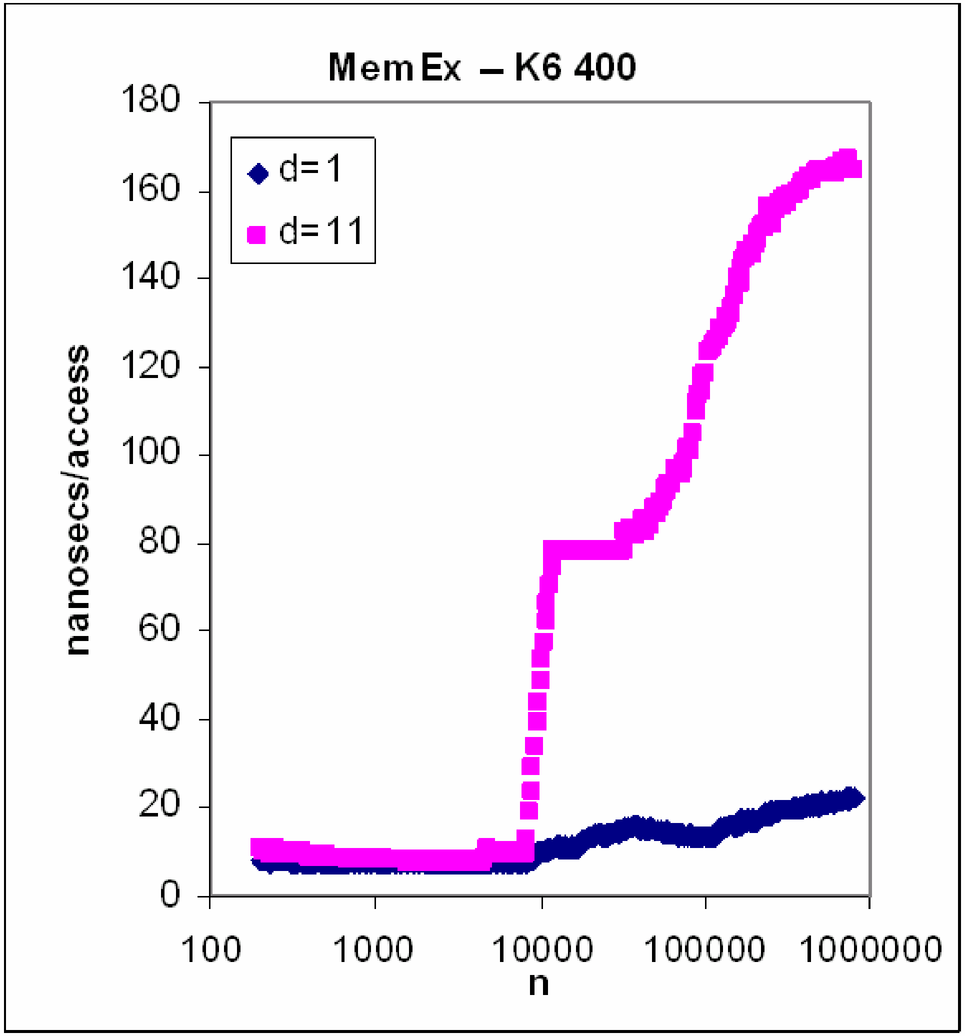
```
for (i = 0; i < count; i++) {
    sum += x[j];
    j += d;
    if (j >= n)
        j -= n;
}
```

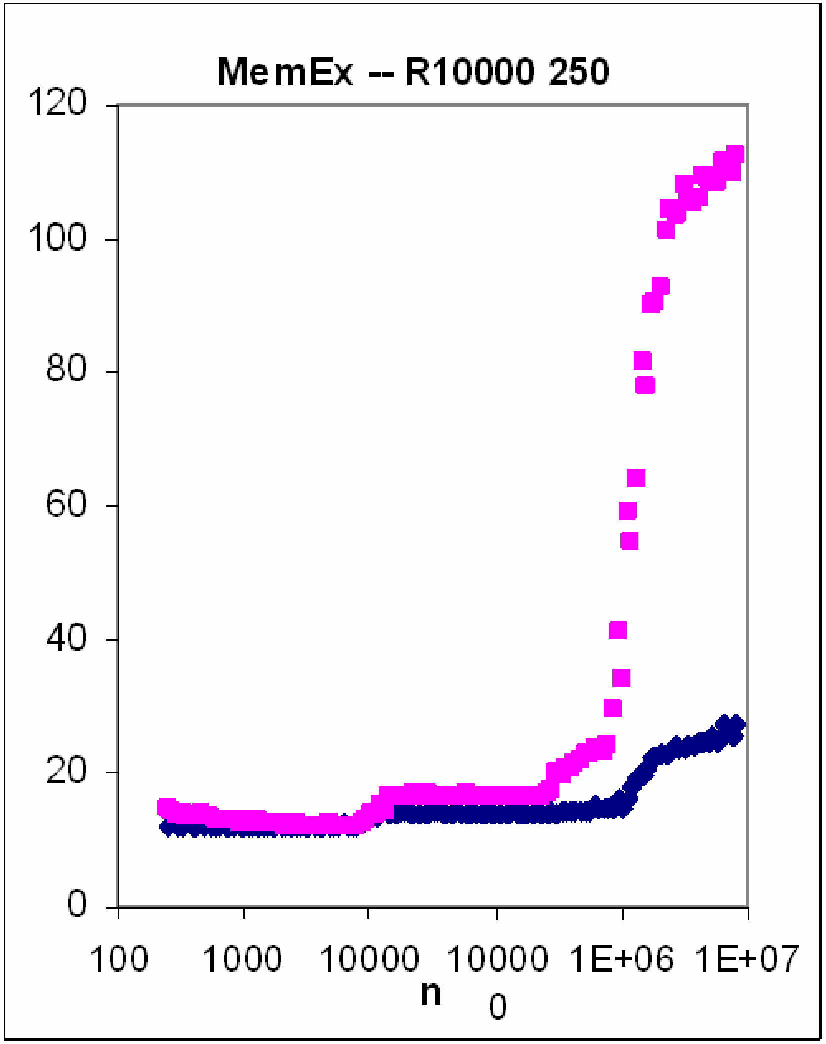
The complete MemEx program is ~30 lines of C

Results of the Model



Other Machines





An insight that can explain *and* predict.

Simon's Parable of the Ant

"We watch an ant make his laborious way across a wind- and wave molded beach. He moves ahead, angles to the right to ease his climb up a steep dunelet, detours around a pebble, …. I sketch his path on a piece of paper….

"I show the unlabeled sketch to a friend. Whose path is it? An expert skier, perhaps, slaloming down a steep and somewhat rocky slope. Or a sloop, beating upwind in a channel dotted with islands or shoals….

"An ant, viewed as a behaving system, is quite simple. The apparent complexity of its behavior over time is largely a reflection of the complexity of the environment in which it finds itself."

MemEx is a simple program in a complex environment

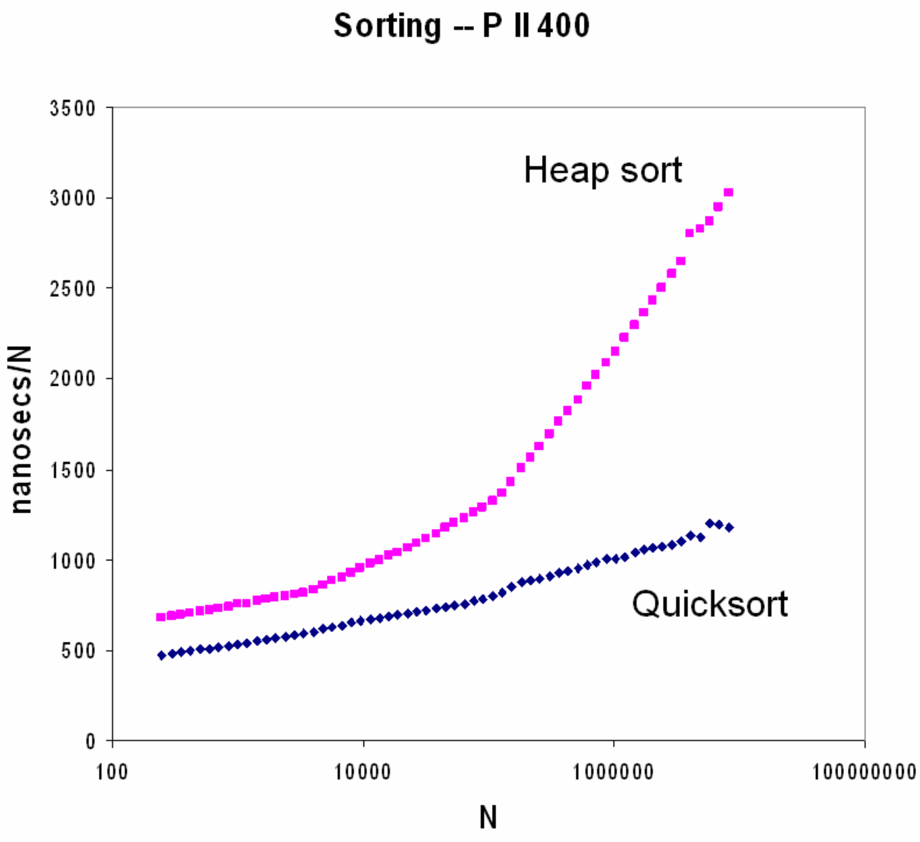
Statistical Questions

- How can I {explain, formalize, reason about} experiments across memory domains?
  - How can I reason about large experiments built on this infrastructure?
    - Tukey, *EDA*, 6C: "We can make many good uses of a close fit, whether or not it is 'a basic law'."
  - How can I talk about the components of a computation?
- How can I design experiments to get the most insight bang for the least computational buck?
  - This tiny experiment did well; how should I design an experiment on cell-based NN searching?
  - How can I design algorithms for this environment?
- How ought I deal with hiccups?

Statements about Sorting?

In memory systems with equal costs for random and sequential access, this Heap sort is about 30%-50% slower than this Quicksort

In memory systems with random access a factor of *K* more expensive than sequential access, the dominant term of this Heap sort is about a factor of 1.3*K* greater than the dominant term of this Quicksort



A Tale of Two Sorts

Heap sort is O(n log n) in the worst case

- **Pretty fast on the average**

Quicksort is expected O(n log n)

- **Faster yet on the average (30% or more)**
- **Frequently used to implement the C `qsort`**
- **$\Theta(n^2)$ in the worst case**

A Great Bug Report

We [Wilks & Becker] found that `qsort` is unbearably slow on "organ-pipe" inputs like "0123443210":

```
main(int argc, char **argv)
{   int n=atoi(argv[1]), i, x[100000];
    for (i = 0; i < n; i++)
        x[i] = i;
    for ( ; i < 2*n ; i++)
        x[i] = 2*n-i-1;
    qsort(x, 2*n, sizeof(int), intcmp);
}
```

(Continued …)

Wilks and Becker, Cont.

Here are the timings on a Pentium:

```
$ time a.out 2000
real    5.85s
$ time a.out 4000
real    21.65s
$ time a.out 8000
real    85.11s
$
```

This is clearly quadratic behavior – each time we double the input size, the run time goes up by a factor of four.

A simple experiment to reveal that a sort that should be O(*n* log *n*) is in fact quadratic

- Distilled from a huge program (hundreds of thousands of lines of code)

Where Does the Time Go?

Observation

- **The run time of the little program is quadratic**

Explanations?

- **A flaw in qsort itself**
- **Expensive underlying structures**
  - **Memory management: caching?**
  - **Memory allocation: malloc?**

A Production Malloc

A malloc driver

```
void main(int argc, char *argv[])
{   int n = atoi(argv[1]), m = atoi(argv[2]);
    while (n-- > 0)
        malloc(m);
}
```

Some runs for 16-byte nodes

```
$ time a.out 50000 16
1.8u
$ time a.out 100000 16
8.5u
$ time a.out 200000 16
38.3u
$
```

McIlroy's explanation

How Firm a Foundation?

- Bumps so far
  - Memory hierarchy
    - Caching, paging
  - Memory management
    - Storage allocation
- Additional traps lurking
  - Software: Compiler optimizations
  - Hardware: Deep pipelines
  - Multiprocessing, Networks, Parallel computation, …
- Simon’s Parable of the Ant suggests that simple experiments may provide a powerful way to explore complex environments

Return to Qsort

Becker and Wilks observed quadratic CPU time

- Many potential sources

Critical operations

- Comparisons and swaps

Add to the code

```
int c = 0;

int intcmp (int *i, int *j)
{   c++; return *i - *j; }

printf("comps: %d\n", c);
```

A Hypothesis About Qsort

A sequence of runs

```
$ a.out 1000
comps: 1000000
$ a.out 2000
comps: 4000000
$ a.out 4000
comps: 16000000
```

1. Extrapolation
3. Robust Conclusions
5. Components – comps or overhead?

Next Steps

Why n^2 time to sort $2n$ elements?

A better `qsort`

String Reversal in Awk

How do algorithms on English words perform right-to-left, rather than left-to-right?

1.) Rewrite a suite of programs

2.) Reverse the words in the English input files

A "Production" Program

A quick and insightful experiment

```
function rev1(s,  len, i, t) {
        len = length(s)
        t = ""
        for (i = 1; i <= len; i++)
                t = substr(s, i, 1) t
        return t
}
{       print rev1($1)
}
```

abcd →
<null>
a
ba
cba
dcba

A "Theoretical" Question

How much time to reverse a string?

Is there a better way?

Faster String Reversal?

A Divide-and-Conquer Algorithm

```
function rev2(s,   len, m) {
     len = length(s)
     if (len <= 1)
          return s
     m = int(len/2)
     return rev2(substr(s, m+1)) \
            rev2(substr(s, 1, m))
}
```

abcdefgh →
efgh abcd
gh ef cd ab
hg fe dc ba

Questions

Correct?

How fast compared to `rev1`?

Summary of Reversal

Data

Alg	N	Secs
1	64,000	2.53
1	128,000	11.38
2	256,000	3.51
2	512,000	6.99

Hypotheses

- Alg 1 is quadratic
- Alg 2 is O(*N* log *N*)
- Comparing Algs 1 and 2   A tiny “horse race”
  - Alg 2 is faster (for values of *N* in this neighborhood)
  - Alg 1 takes 7 lines of code; Alg 2 takes 8 lines

Complete Experimental Apparatus

```
awk '
function rev1(s,  len, i, t) {
    len = length(s)
    t = ""
    for (i = 1; i <= len; i++)
        t = substr(s, i, 1) t
    return t
}

function rev2(s,   len, m) {
    len = length(s)
    if (len <= 1)
        return s
    m = int(len/2)
    return rev2(substr(s, m+1)) \
           rev2(substr(s, 1, m))
}

  { alg = $1
    n = $2
    s = "a"
    while (length(s) < n)
        s = s s
    if (alg == 1)
        s = rev1(s)
    else
        s = rev2(s)
    print alg, length(s)
  }
' $*
```

```
time reverse <<End
1 64000
End
time reverse <<End
1 128000
End
time reverse <<End
2 256000
End
time reverse <<End
2 512000
End
```

```
2.533
11.376
3.515
6.989
```

1. Extrapolation
2. Comparisons
3. Robust Conclusions
5. Components (string costs)

Compressing American Names

- General Idea: Store unique large items in a master table, and then represent them elsewhere by small indices into that table
- Application to Last Names
  - Idea
    - Represent Smith by 1, Johnson by 2, etc.
  - Performance
    - Are names nonuniform?
      - How often do frequent names appear?
    - How to phrase a precise question?

Data on Last Names

From `www.census.gov/genealogy/names`

```
SMITH      1.006  1.006 1
JOHNSON    0.810  1.816 2
WILLIAMS   0.699  2.515 3
  …
HOLLAND    0.042 27.786 256
  …
BOBECK     0.000 88.093 65536
```

Top name accounts for 1% of the 1990 US population

Top 256 names account for 28% of the population

Top 65,536 names account for 88% of the population

1 byte encodes 28%; 2 bytes encode 88%

This data is not dispositive, but may be typical

Data on Male First Names

From `www.census.gov/genealogy/names`

JAMES	3.318	3.318	1
JOHN	3.271	6.589	2
ROBERT	3.143	9.732	3
…			
GEORGE	0.927	28.939	16
…			
DWAYNE	0.059	76.480	256

Top name accounts for 3.3% of the population

Top 16 names account for 29% of the population

Top 256 names account for 76% of the population

1 nybble encodes 29%; 1 byte encodes 76%

Where Does The Time Go?

A Built-In Profiler

- Detailed counts on the time spent in each function give the complete distribution

Dunlavey's Call-Stack Sampling

- Run the program under a debugger, halt it with a "pause" button, and examine the call stack. Make a record of the call stacks observed.
- Any statement that appears on more than one call stack might be a time hog.
- Invoking a statement less frequently (or eliminating it) reduces execution time by the fraction of time it resided on the call stack.
- [Details in SIGPLAN Notices and Wikipedia]

Ancient History

Programming Pearls, "The Back of the Envelope"

"Bob Martin read from a proposal for the system that his team was building for the Olympic games, and went through a similar sequence of calculations.

He estimated one key parameter as we spoke by measuring the time required to send himself a one-character piece of mail.

His calculations showed that, under generous assumptions, the proposed system could work only if there were at least a hundred and twenty seconds in each minute."

A Huge System

How long to send e-mail to 100,000 recipients?

Time to send to *N* recipients:

N	**Seconds**
1000	**11.4**
2000	**39.6**
4000	**167.1**

Tukey, *EDA*, Section 6B

“Three points can take us a long way. If they are well chosen, they can do very well for us.”

Simon on Soft Science

"If I ever believed in the myth of the 'exact sciences' or 'hard sciences', my belief was wholly dissipated by encounters with such topics as air quality, eutrophication of lake waters, global warming, dietary standards, effects of low-level radiation, meteorology (for example, cloud seeding), and cold fusion. All of these topics contain uncertainties about the facts and their implications at least as serious as those we are accustomed to in the social sciences.

The true line is not between 'hard' natural science and 'soft' social sciences, but between

- precise science limited to highly abstract and simple phenomena in the laboratory and 1968: MIX 1009
- inexact science and technology dealing with complex phenomena in the real world." 2008: microprocessors

H. A. Simon, *Models of My Life*, p. 304

The Meaning of “Little”

- Two Definitions
  - A few slides to describe
  - Less than an hour to conduct
- Counting Time
  - Conducting the final experiment
    - From scratch: Compressing names, string reversal, MemEx, Dunlavey’s sampling
    - From working code: Cell NN searching, sorting
    - After days of debugging: Qsort, malloc
    - After organizational hurdles: E-mail system
  - A new experiment in an existing framework
    - Next two stories: bin packing, k-d trees

History: Analysis of K-d Trees

K-d trees

Nearest Neighbor search

13 August 1974

13 Aug 1974

Empirical Test 13 Aug 1974 – 25 Trees/size

n	$\log_2 n$	V_n
10	3.32193	6.7276
20	4.32193	9.3644
30	4.90689	11.1932
40	5.32193	11.9012
50	5.64386	12.9380
60	5.90689	13.8568
70	6.12929	14.1216
80	6.32193	14.7944
90	6.49184	14.7188
100	6.64386	15.0640
200	7.64386	17.6016
400	8.64386	20.2868
600	9.22881	21.5396
800	9.64386	21.9680
1000	9.96578	23.2984
1200	10.22861	23.2944
1400	10.45122	23.7764
1600	10.64386	24.4592
1800	10.81377	24.6660
2000	10.96578	24.9256
2200	11.10328	25.6508
2400	11.22881	25.6344
2600	11.34428	25.5188

Trim the data

The power of graphical displays

Any graph at all

A graph of the right variable

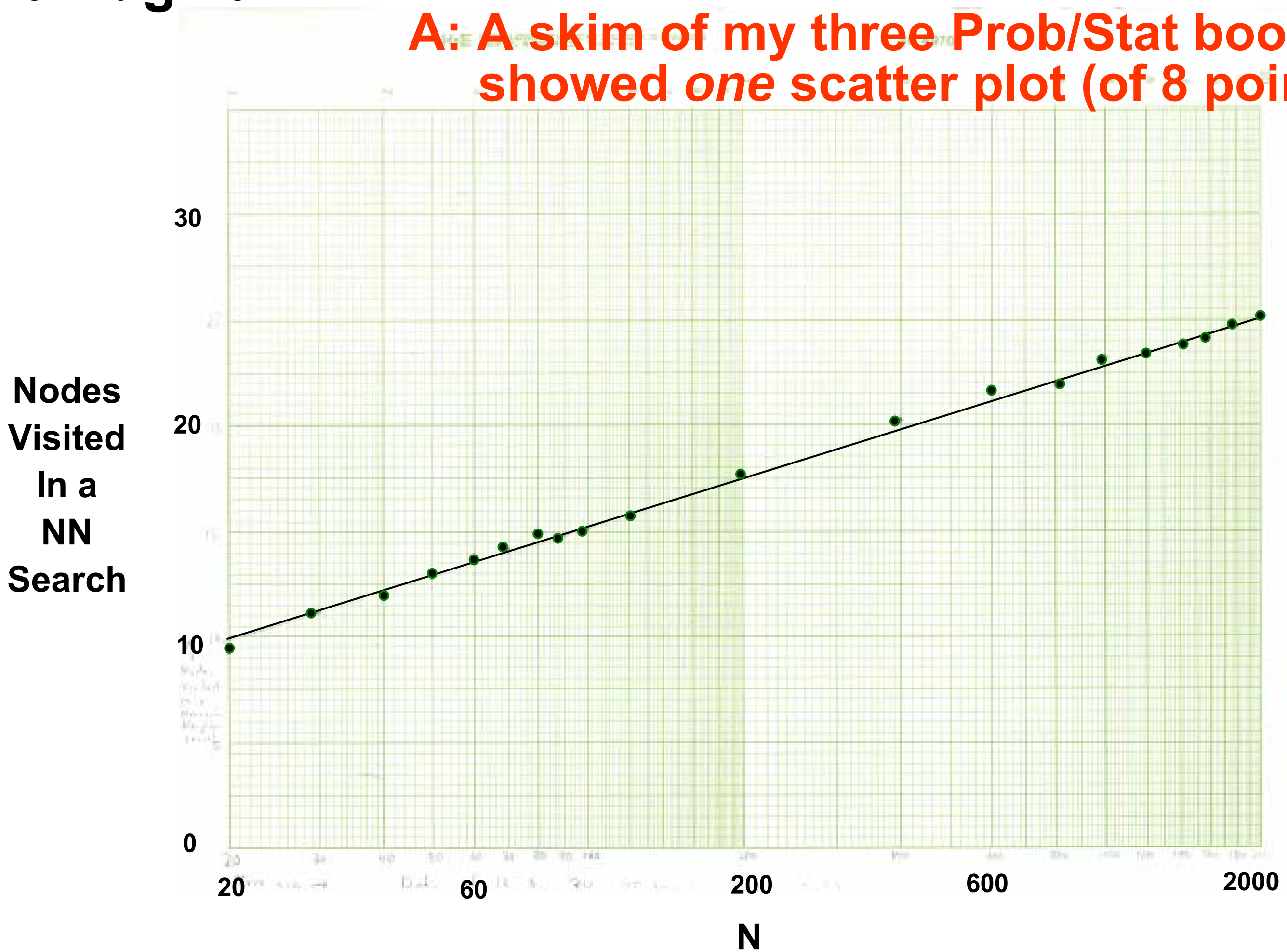
An Old Graph

Nearest Neighbor search in K-d trees

13 Aug 1974

Q: Why did it take me so long to draw this graph?

A: A skim of my three Prob/Stat books showed *one* scatter plot (of 8 points!)



Why the bumps?

Later Graphs

1. Extrapolation
5. Components of a count
3. Robust Conclusions

Two separable issues

Work going down the tree: node visits

Work at the leaves: dist calcs in the buckets

The Whole Truth

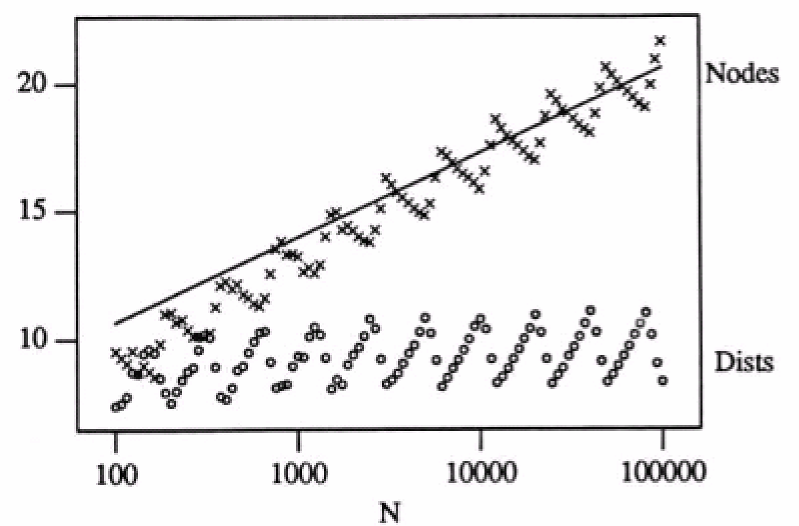


Figure 5.1. All nearest neighbors, top-down search, bucket cutoff 5.

Observations: tradeoffs and cyclicity (binary trees)

A Peek at Asymptopia

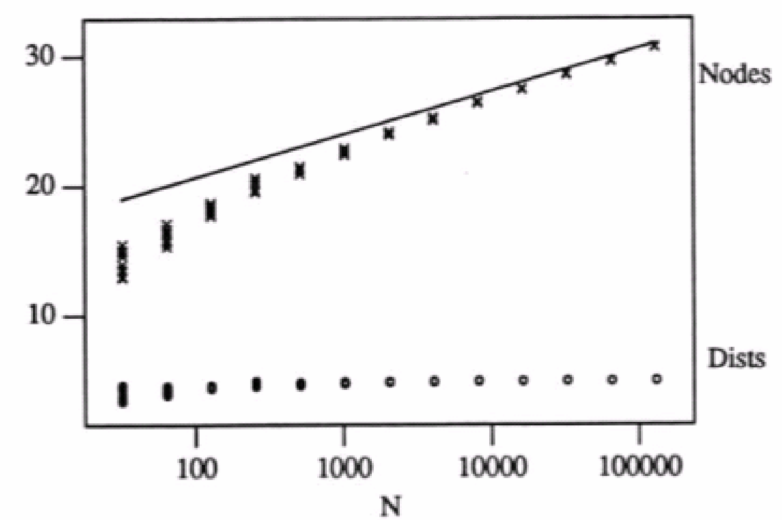


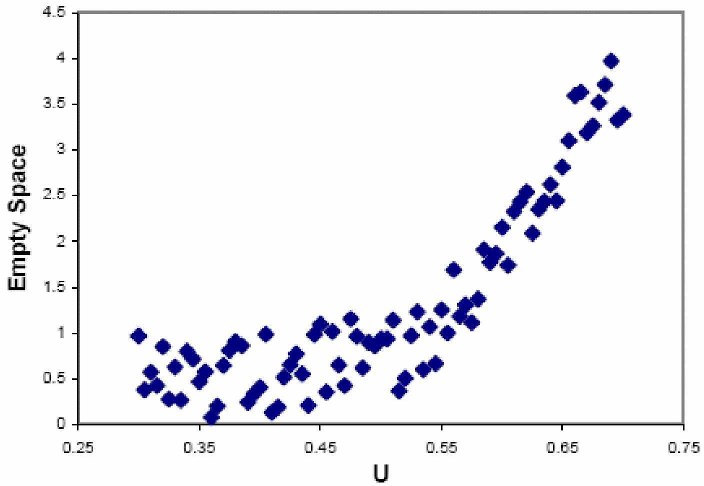
Figure 5.2. All nearest neighbors, top-down search, bucket cutoff 1.

Plea for statistical help: How to design this experiment?

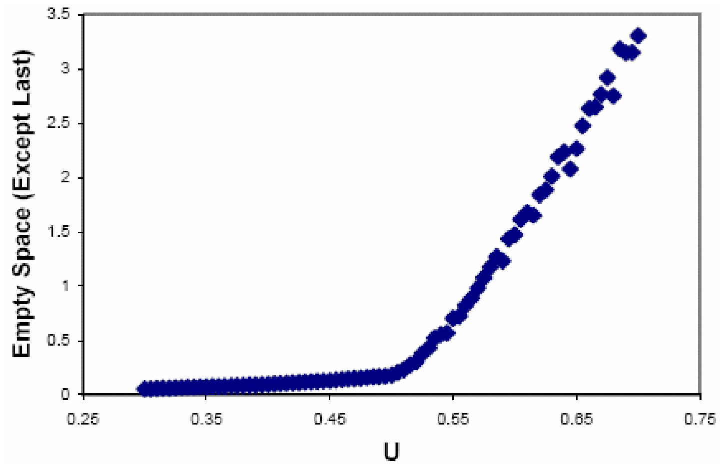
Graphs of Bin Packing

1. Extrapolation
3. Robustness
5. Components

Graph 1: Empty space, *n* = 128,000

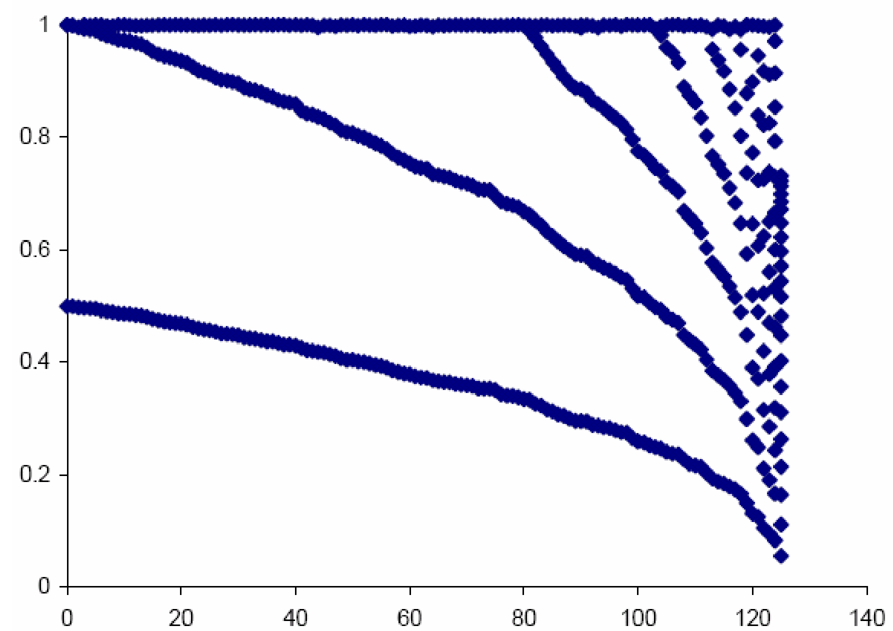


Graph 2: Empty except last



Observation: spread is about 1

A single packing of bins (l-to-r)



Insight: ignore the final bin!

Zoo of Little Experiments

NN searching with cells	1980
Sort times under caching	1999
Memory cost model	2000
A broken qsort	1991
An expensive malloc	1995
String reversal in Awk	2004
Compressing USA names	2006
Dunlavey's sampling	2007
An e-mail system	2007
NN searching in k-d trees	1974, 1991
Bin packing	1983, 2005

A third of a century of inducing intraocular trauma

Sampler of colleagues: Friedman, Weide, Yao, Leighton, Johnson, McGeoch, Kernighan, McIlroy, …

Context: Tiny MSE

Problem from colleagues

- Hashing ~20,000 100-character strings into 32-bit ints
  - Too many collisions?
  - How many should we expect?
- Birthday problem — Math
  - With 23 people, about a 50% chance of shared birthdays
  - When tossing balls into *N* urns, probability of 50% of shared urn after about $N^{1/2}$ tosses
- Observe about 10 collisions after 100 hashes

Solution

Science

- A tiny horse race: testbed to count collisions for several hash functions
- A couple hours later, the best was “good enough”

Engineering

Kinds of Experiments

Parameter Estimation

NN searching with cells
Sort times under caching
Memory cost model
String reversal in Awk
Compressing USA names
Bin packing
E-mail system

Functional Form

Sort times under caching
Memory cost model
Costly qsort, malloc
String reversal in Awk
E-mail system
NN searching in k-d trees
Bin packing

Hypothesis Testing

Sort times under caching
Memory cost model
Costly qsort, malloc

Horse Races

NN searching with cells
Sort times under caching
String reversal in Awk

Lessons

Reduce a huge problem to its tiny essence

- Theory: NN searching, bin packing
- Practice: Broken qsort and malloc; e-mail system

A fundamental iteration

- Measure run time ⇔ count key operations

Identify critical environmental issues

- Caching, malloc, qsort, etc.

Draw graphs

- Plot the right variables
- Look at enough detail to expose the *real* shape

Think small

Why Small Experiments on Algorithms?

- **Small is Cheap**
  - **Useful for making engineering decisions**
    - **Dunlavey's sampling**
- **Small is Insightful**
  - **Results are usually straightforward to interpret**
  - **Decompose a big system into critical components**
    - **Memory cost model**
- **Small is Beautiful**
  - **Document serious flaws in real systems**
    - **Qsort, malloc, e-mail**
  - **New insights for bin packing and *k*-d trees**

Plea for Statistical Help

1. Statistical common sense
   - Few-point analyses: revise the lost art of *EDA*
   - Drawing simple graphs
     - Most of these graphs are cost as a function of size

More substantial issues

2. How do I reason about layered systems (such as caching)?
   - What canonical tests must I perform?
   - What precise statements do I then make?
3. How do I design experiments to get the most insight bang for the least computational buck?
   - Neil Sloane: A *sequence* of experiments – given goals and results of experiments 1 … *N*, how to design *N*+1?

Special Bonus Material!

Probably not time in the workshop

A Normal Form for describing experiments

Feedback

- Algorithms people: How can we describe little experiments?
- Statisticians: How do you adress similar problems?

A Small Medical Experiment

Lesson 2: Format

Gunn, et al. "How should an unconscious person with a suspected neck injury be positioned?" *Prehospital Disaster Med.* 1995 Oct-Dec;10(4):239-44.

Hypothesis. It is proposed that the HAINES (High Arm IN Endangered Spine) modified recovery position reduces movement of the neck. In this modification, one of the patient's arms is raised above the head to support the head and neck.

Methods. Neck movements in two healthy volunteers were measured by video-image analysis and radiographic studies.

Results. For both subjects, the total degree of lateral flexion of the cervical spine in the HAINES position was less than half of that measured during use of the lateral recovery position.

Conclusion. An unconscious person with a suspected neck injury should be positioned in the HAINES modified recovery position. There is less neck movement than when the lateral recovery position is used, and, therefore, HAINES use carries less risk of spinal-cord damage.

Lesson 1: Size – 2002: 38 subjects, more analysis

Sort Algorithms in the Presence of Caching

Background

Domain: Sorting algorithms

Motivation: Determine the effect of caching on the run time of sort algorithms

Goal: A simple model to predict performance under caching

Algorithm

Design techniques: Implement existing Quicksort and heapsort algorithms

Code: ~50 lines of C

Experimental Apparatus

Overview: Single program implementing driver and algs

Generated Input Data: Arrays of uniformly distributed integers

Output: Run times of sorts

Analysis

Technique: Graph time per element as a function of size (log scale)

Results: Without caching, plots would be linear. With caching, linear within L1, L2 and RAM

Conclusions: Simple model explains caching times

A Slow Malloc

Background

Domain: C library malloc function

Motivation: Production library malloc was flawed

Goal: Document that the library malloc is unreasonably slow

Algorithm

Name: Unknown storage allocation algorithm

Design techniques: Unknown

Code: ~5 lines of driver + library malloc

Experimental Apparatus

Overview: Simple driver to malloc *n* blocks each of size *m*

Generated Input Data: None

Output: Run times via time(1)

Analysis

Technique: Few-point analysis

Results: Run time is quadratic for a simple class of inputs

Conclusions: A system library function should be rewritten

References: "With malloc aforethought"

A Slow Qsort

Background

Domain: C library qsort function

Motivation: Production library sort was flawed

Goal: Document that the library sort is unreasonably slow

Algorithm

Name: Quicksort implementation of C qsort interface

Design techniques: Divide-and-conquer

Code: ~10 lines of driver + ~100 lines of broken sort

Experimental Apparatus

Overview: Simple driver to generate one input and call qsort

Generated Input Data: "Organ pipe" input arrays of the form 012345543210

Output: Run times via time(1)

Analysis

Technique: Few-point analysis

Results: Run time is quadratic for a plausible class of inputs

Conclusions: A system library function should be rewritten

References: "The trouble with Qsort", "Engineering a sort function"

String Reversal in Awk

Background

Domain: Efficient string operations in a high-level language

Motivation: Fundamental problem; useful algorithm design techniques; cost models

Goal: Confirm that the cost of a straightforward string reversal algorithm is quadratic, and design a more efficient algorithm

Algorithm

Design techniques: Iteration, divide-and-conquer

Code: ~30 lines of Awk

Experimental Apparatus

Overview: Single program to implement and exercise two algorithms

Generated Input Data: Strings of the form a^N

Output: Run times via time(1)

Analysis

Technique: Few-point analysis

Results: Simple algorithm is apparently $\Theta(N^2)$, while divide-and-conquer algorithm is $\Theta(N \lg N)$

Conclusions: Divide-and-conquer can be simple and useful

Representing Common Names

Background

Domain: Data compression

Motivation: Represent a directory in an 8MB telephone

Goal: Represent a set of names in as little space as possible

Algorithm

Design techniques: Replacing a string by an index into a table of strings

Code: None; measurement only

Experimental Apparatus

Overview: Examine publicly available data on distribution of USA first and last names

Analysis

Technique: Few-point analysis

Results: For American last names, one byte represents 28% and two bytes represents 88% of the population. For American male first names, one nybble represents 29%, and one byte represents 76%.

Conclusions: Simple techniques could provide effective compression